



TITLE:

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Algebraic independence of the values of a certain map defined on the set of orbits of the action of Klein four-group

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1 Introduction

Let $\{R_k\}_{k \geq 1}$ be a linear recurrence of positive integers satisfying

$$R_{k+n} = c_1 R_{k+n-1} + \cdots + c_n R_k \quad (k \geq 1), \quad (1)$$

where $n \geq 2$ and c_1, \dots, c_n are nonnegative integers with $c_n \neq 0$. The author [9] studied the two-variable function $E(x, q)$ defined by

$$E(x, q) = \sum_{k=1}^{\infty} \prod_{l=1}^k \frac{xq^{R_l}}{1 - q^{R_l}} = \sum_{k=1}^{\infty} \frac{x^k q^{R_1 + R_2 + \cdots + R_k}}{(1 - q^{R_1})(1 - q^{R_2}) \cdots (1 - q^{R_k})},$$

which may be regarded as an analogue of q -exponential function

$$E_q(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k q^{1+2+\cdots+k}}{(1 - q)(1 - q^2) \cdots (1 - q^k)}$$

(cf. Gasper and Rahman [2]), if we replace k in the exponent of q in $E_q(x)$ with $\{R_k\}_{k \geq 1}$ defined above.

Let

$$\Phi(X) = X^n - c_1 X^{n-1} - \cdots - c_n \quad (2)$$

and let $\overline{\mathbb{Q}}^\times$ be the set of nonzero algebraic numbers. The author proved the following

Theorem 0 (Corollary 4 of [9]). *Let $\{R_k\}_{k \geq 1}$ be a linear recurrence satisfying (1). Suppose that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Assume that $\{R_k\}_{k \geq 1}$ is not a geometric progression. Then the values*

$$E(x, q) \quad (x, q \in \overline{\mathbb{Q}}^\times, |q| < 1)$$

are algebraically dependent if and only if there exist some distinct pairs (x_1, q_1) and (x_2, q_2) of nonzero algebraic numbers x_1, x_2, q_1 , and q_2 with $|q_1|, |q_2| < 1$ such that $x_1 = x_2$ and $q_1^{N_k} = q_2^{N_k}$ for some $k \geq 1$, where $N_k = \text{g.c.d.}(R_k, R_{k+1}, \dots, R_{k+n-1})$.

In particular, if $N_k = 1$ for any $k \geq 1$, then the values $E(x, q)$ are algebraically independent for any distinct pairs (x, q) of nonzero algebraic numbers x and q with $|q| < 1$.

Example 0. Let $\{F_k\}_{k \geq 1}$ be the sequence of Fibonacci numbers defined by $F_1 = 1$, $F_2 = 1$, and $F_{k+2} = F_{k+1} + F_k$ ($k \geq 1$). Since $\{F_k\}_{k \geq 1}$ satisfies the conditions in Theorem 0, the infinite set of the values

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{F_1+F_2+\dots+F_k}}{(1-q^{F_1})(1-q^{F_2})\dots(1-q^{F_k})} \mid x, q \in \overline{\mathbb{Q}}^\times, |q| < 1 \right\}$$

is algebraically independent.

The two-variable function $E(x, q)$ converges on the domain

$$(\mathbb{C} \times \{|q| < 1\}) \cup (\{|x| < 1\} \times \{|q| > 1\}) := \{(x, q) \in \mathbb{C}^2 \mid |q| < 1 \vee (|x| < 1 \wedge |q| > 1)\},$$

whereas a ‘balanced’ analogue

$$\Theta(x, q) = \sum_{k=1}^{\infty} \prod_{l=1}^k \frac{x q^{R_l}}{1 - q^{2R_l}} = \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\dots+R_k}}{(1 - q^{2R_1})(1 - q^{2R_2})\dots(1 - q^{2R_k})}$$

converges on the wider domain

$$\mathbb{C} \times \{|q| \neq 1\} := \{(x, q) \in \mathbb{C}^2 \mid |q| \neq 1\}.$$

Indeed, if $q \neq 0$, $\Theta(x, q)$ is invariant under the map

$$\sigma_1 : (x, q) \mapsto (-x, q^{-1}),$$

namely

$$\Theta(\sigma_1(x, q)) = \sum_{k=1}^{\infty} \frac{(-x)^k q^{-R_1-R_2-\dots-R_k}}{(1 - q^{-2R_1})(1 - q^{-2R_2})\dots(1 - q^{-2R_k})} = \Theta(x, q)$$

and so $\Theta(x, q)$ converges on $\mathbb{C} \times \{|q| \neq 1\}$ by the similar reason to the convergence of $E(x, q)$.

Moreover, if $\{R_k\}_{k \geq 1}$ is a sequence of odd integers, then $\Theta(x, q)$ is invariant also under the maps

$$\begin{aligned} \sigma_2 &: (x, q) \mapsto (-x, -q), \\ \sigma_3 &: (x, q) \mapsto (x, -q^{-1}). \end{aligned}$$

Since $\sigma_1 \circ \sigma_1 = \sigma_2 \circ \sigma_2 = \text{id}$ and $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1 = \sigma_3$, we see that $G_4 = \{\text{id}, \sigma_1, \sigma_2, \sigma_3\}$ is Klein four-group. Therefore, $\Theta(x, q)$ can be regarded as a map defined on the set of orbits $(\mathbb{C} \times \{|q| \neq 0, 1\})/G_4$, where $\mathbb{C} \times \{|q| \neq 0, 1\} = \{(x, q) \in \mathbb{C}^2 \mid |q| \neq 0, 1\}$, namely the map

$$\tilde{\Theta} : (\mathbb{C} \times \{|q| \neq 0, 1\})/G_4 \longrightarrow \Theta(\mathbb{C} \times \{|q| \neq 0, 1\})$$

given by

$$\text{the orbit of } (x, q) \mapsto \Theta(x, q)$$

is well-defined. Hence the restriction to algebraic points

$$\tilde{\Theta} : \left((\mathbb{C} \times \{|q| \neq 0, 1\}) \cap (\overline{\mathbb{Q}}^\times)^2 \right) / G_4 \longrightarrow \Theta \left((\mathbb{C} \times \{|q| \neq 0, 1\}) \cap (\overline{\mathbb{Q}}^\times)^2 \right),$$

or equivalently

$$\tilde{\Theta} : \left(\overline{\mathbb{Q}}^\times \times (\overline{\mathbb{Q}}^\times \setminus \{|q| = 1\}) \right) / G_4 \longrightarrow \Theta \left(\overline{\mathbb{Q}}^\times \times (\overline{\mathbb{Q}}^\times \setminus \{|q| = 1\}) \right)$$

is also well-defined, where the second $\overline{\mathbb{Q}}^\times$ denotes the multiplicative group of nonzero algebraic numbers while the first $\overline{\mathbb{Q}}^\times$ simply denotes the set of nonzero algebraic numbers. In this paper we prove the following

Theorem 1. *Let $\{R_k\}_{k \geq 1}$ be a linear recurrence satisfying (1). Suppose that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Assume that $\text{g.c.d.}(R_k, R_{k+1}, \dots, R_{k+n-1}) = 1$ for any $k \geq 1$. Assume further that $\Phi(2) < 0$ and that $\{R_k\}_{k \geq 1}$ is a sequence of odd integers. Then the infinite set of the values*

$$\tilde{\Theta} \left(\left(\overline{\mathbb{Q}}^\times \times (\overline{\mathbb{Q}}^\times \setminus \{|q| = 1\}) \right) / G_4 \right)$$

is algebraically independent.

Remark 1. The condition that $\text{g.c.d.}(R_k, R_{k+1}, \dots, R_{k+n-1}) = 1$ for any $k \geq 1$ implies that the sequence $\{R_k\}_{k \geq 1}$ is not a geometric progression.

Corollary 1. *Let $\{R_k\}_{k \geq 1}$ be as in Theorem 1. Then the infinite set consisting of the distinct values of*

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\dots+R_k}}{(1-q^{2R_1})(1-q^{2R_2})\dots(1-q^{2R_k})} \mid x, q \in \overline{\mathbb{Q}}^\times, |q| \neq 1 \right\}$$

is algebraically independent.

Example 1. Let $\{P_k\}_{k \geq 1}$ be the sequence defined either by $P_1 = P_2 = 1$ and $P_{k+2} = 2P_{k+1} + P_k$ ($k \geq 1$) or by $P_1 = P_2 = P_3 = 1$ and $P_{k+3} = P_{k+2} + P_{k+1} + 3P_k$ ($k \geq 1$). Since $\{P_k\}_{k \geq 1}$ satisfies all the conditions of Theorem 1, the infinite set consisting of the distinct values of

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{P_1+P_2+\dots+P_k}}{(1-q^{2P_1})(1-q^{2P_2})\dots(1-q^{2P_k})} \mid x, q \in \overline{\mathbb{Q}}^\times, |q| \neq 1 \right\}$$

is algebraically independent.

If $\{R_k\}_{k \geq 1}$ is a sequence of odd integers, then

$$\Theta_+(x, q) := \sum_{k=1}^{\infty} \prod_{l=1}^k \frac{xq^{R_l}}{1+q^{2R_l}} = \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\dots+R_k}}{(1+q^{2R_1})(1+q^{2R_2})\dots(1+q^{2R_k})}$$

is invariant under the maps

$$\begin{aligned} \tau_1 &: (x, q) \longmapsto (x, q^{-1}), \\ \tau_2 &: (x, q) \longmapsto (-x, -q), \\ \tau_3 &: (x, q) \longmapsto (-x, -q^{-1}). \end{aligned}$$

Since $\tau_1 \circ \tau_1 = \tau_2 \circ \tau_2 = \text{id}$ and $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1 = \tau_3$, we see that $G'_4 = \{\text{id}, \tau_1, \tau_2, \tau_3\}$ is also Klein four-group. Hence the map

$$\tilde{\Theta}_+ : (\mathbb{C} \times \{|q| \neq 0, 1\})/G'_4 \longrightarrow \Theta_+(\mathbb{C} \times \{|q| \neq 0, 1\})$$

given by

$$\text{the orbit of } (x, q) \longmapsto \Theta_+(x, q)$$

is well-defined. We also have the following

Theorem 2. *Let $\{P_k\}_{k \geq 1}$ be as in Theorem 1. Then the infinite set of the values*

$$\tilde{\Theta}_+ \left(\left(\overline{\mathbb{Q}}^\times \times (\overline{\mathbb{Q}}^\times \setminus \{|q| = 1\}) \right) / G'_4 \right)$$

is algebraically independent.

Example 2. Let $\{P_k\}_{k \geq 1}$ be one of the linear recurrences defined in Example 1. Since $\{P_k\}_{k \geq 1}$ satisfies all the conditions of Theorem 1, the infinite set consisting of the distinct values of

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{P_1+P_2+\dots+P_k}}{(1+q^{2P_1})(1+q^{2P_2})\dots(1+q^{2P_k})} \mid x, q \in \overline{\mathbb{Q}}^\times, |q| \neq 1 \right\}$$

is algebraically independent.

2 Lemmas

Let $F(z_1, \dots, z_n)$ and $F[[z_1, \dots, z_n]]$ denote the field of rational functions and the ring of formal power series in variables z_1, \dots, z_n with coefficients in a field F , respectively, and F^\times the multiplicative group of nonzero elements of F . Let $\Omega = (\omega_{ij})$ be an $n \times n$ matrix with nonnegative integer entries. Then the maximum ρ of the absolute values of the eigenvalues of Ω is itself an eigenvalue (cf. Gantmacher [1, p. 66, Theorem 3]). If $\mathbf{z} = (z_1, \dots, z_n)$ is a point of \mathbb{C}^n , we define a transformation $\Omega : \mathbb{C}^n \rightarrow \mathbb{C}^n$ by

$$\Omega \mathbf{z} = \left(\prod_{j=1}^n z_j^{\omega_{1j}}, \prod_{j=1}^n z_j^{\omega_{2j}}, \dots, \prod_{j=1}^n z_j^{\omega_{nj}} \right). \quad (3)$$

We suppose that Ω and an algebraic point $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, where α_i are nonzero algebraic numbers, have the following four properties:

- (I) Ω is nonsingular and none of its eigenvalues is a root of unity, so that in particular $\rho > 1$.
- (II) Every entry of the matrix Ω^k is $O(\rho^k)$ as k tends to infinity.
- (III) If we put $\Omega^k \boldsymbol{\alpha} = (\alpha_1^{(k)}, \dots, \alpha_n^{(k)})$, then

$$\log |\alpha_i^{(k)}| \leq -c\rho^k \quad (1 \leq i \leq n)$$

for all sufficiently large k , where c is a positive constant.

(IV) For any nonzero $f(\mathbf{z}) \in \mathbb{C}[[z_1, \dots, z_n]]$ which converges in some neighborhood of the origin, there are infinitely many positive integers k such that $f(\Omega^k \boldsymbol{\alpha}) \neq 0$.

Lemma 1 (Lemma 4 and Proof of Theorem 2 in [6]). *Suppose that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity, where $\Phi(X)$ is the polynomial defined by (2). Let*

$$\Omega = \begin{pmatrix} c_1 & 1 & 0 & \dots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ c_n & 0 & \dots & \dots & 0 \end{pmatrix} \quad (4)$$

and let β_1, \dots, β_s be multiplicatively independent algebraic numbers with $0 < |\beta_j| < 1$ ($1 \leq j \leq s$). Let p be a positive integer and put $\Omega' = \text{diag}(\underbrace{\Omega^p, \dots, \Omega^p}_s)$. Then the matrix Ω'

and the point $\boldsymbol{\beta} = (\underbrace{1, \dots, 1}_{n-1}, \beta_1, \dots, \beta_s, \underbrace{1, \dots, 1}_{n-1})$ have the properties (I)–(IV).

Lemma 2 (Kubota [3], see also Nishioka [5]). *Let K be an algebraic number field. Suppose that $f_1(\mathbf{z}), \dots, f_m(\mathbf{z}) \in K[[z_1, \dots, z_n]]$ converge in an n -polydisc U around the origin and satisfy the functional equations*

$$f_i(\mathbf{z}) = a_i(\mathbf{z})f_i(\Omega\mathbf{z}) + b_i(\mathbf{z}) \quad (1 \leq i \leq m),$$

where $a_i(\mathbf{z}), b_i(\mathbf{z}) \in K(z_1, \dots, z_n)$ and $a_i(\mathbf{z})$ ($1 \leq i \leq m$) are defined and nonzero at the origin. Assume that the $n \times n$ matrix Ω and a point $\boldsymbol{\alpha} \in U$ whose components are nonzero algebraic numbers have the properties (I)–(IV) and that $a_i(\mathbf{z})$ ($1 \leq i \leq m$) are defined and nonzero at $\Omega^k \boldsymbol{\alpha}$ for any $k \geq 1$. If $f_1(\mathbf{z}), \dots, f_m(\mathbf{z})$ are algebraically independent over $K(z_1, \dots, z_n)$, then the values $f_1(\boldsymbol{\alpha}), \dots, f_m(\boldsymbol{\alpha})$ are algebraically independent.

In what follows, C denotes a field of characteristic 0. Let $L = C(z_1, \dots, z_n)$ and let M be the quotient field of $C[[z_1, \dots, z_n]]$. Let Ω be an $n \times n$ matrix with nonnegative integer entries having the property (I). We define an endomorphism $\tau : M \rightarrow M$ by $f^\tau(\mathbf{z}) = f(\Omega\mathbf{z})$ ($f(\mathbf{z}) \in M$) and a subgroup H of L^\times by

$$H = \{g^\tau g^{-1} \mid g \in L^\times\}.$$

Lemma 3 (Kubota [3], see also Nishioka [5]). *Let $f_{ij} \in M$ ($i = 1, \dots, h$; $j = 1, \dots, m(i)$) satisfy*

$$f_{ij} = a_i f_{ij}^\tau + b_{ij},$$

where $a_i \in L^\times$, $b_{ij} \in L$ ($1 \leq i \leq h$, $1 \leq j \leq m(i)$), and $a_i a_{i'}^{-1} \notin H$ for any distinct i, i' ($1 \leq i, i' \leq h$). Suppose for any i ($1 \leq i \leq h$) there is no element g of L satisfying

$$g = a_i g^\tau + \sum_{j=1}^{m(i)} c_j b_{ij},$$

where $c_1, \dots, c_{m(i)} \in C$ are not all zero. Then the functions f_{ij} ($i = 1, \dots, h$; $j = 1, \dots, m(i)$) are algebraically independent over L .

Let $\{R_k\}_{k \geq 1}$ be a linear recurrence satisfying (1) and define a monomial

$$M(\mathbf{z}) = z_1^{R_n} \cdots z_n^{R_1}, \quad (5)$$

which is denoted similarly to (3) by

$$M(\mathbf{z}) = (R_n, \dots, R_1)\mathbf{z}. \quad (6)$$

Let Ω be the matrix defined by (4). It follows from (1), (3), and (6) that

$$M(\Omega^k \mathbf{z}) = z_1^{R_{k+n}} \cdots z_n^{R_{k+1}} \quad (k \geq 0). \quad (7)$$

Lemma 4 (Theorem 2 of [7]). *Suppose that $\{R_k\}_{k \geq 1}$ is not a geometric progression. Assume that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Let \overline{C} be an algebraically closed field of characteristic 0. Suppose that $F(\mathbf{z})$ is an element of the quotient field of $\overline{C}[[z_1, \dots, z_n]]$ satisfying the functional equation of the form*

$$F(\mathbf{z}) = \left(\prod_{k=u}^{p+u-1} Q_k(M(\Omega^k \mathbf{z})) \right) F(\Omega^p \mathbf{z}),$$

where Ω is defined by (4), $p > 0$, $u \geq 0$ are integers, and $Q_k(X) \in \overline{C}(X)$ ($u \leq k \leq p+u-1$) are defined and nonzero at $X = 0$. If $F(\mathbf{z}) \in \overline{C}(z_1, \dots, z_n)$, then $F(\mathbf{z}) \in \overline{C}$ and $Q_k(X) \in \overline{C}^\times$ ($u \leq k \leq p+u-1$).

We adopt the usual vector notation, that is, if $I = (i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n$ with $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers, we write $\mathbf{z}^I = z_1^{i_1} \cdots z_n^{i_n}$. We denote by $C[z_1, \dots, z_n]$ the ring of polynomials in variables z_1, \dots, z_n with coefficients in C .

Lemma 5 (Lemma 3.2.3 in Nishioka [5]). *If $A, B \in C[z_1, \dots, z_n]$ are coprime, then $\text{g.c.d.}(A^\tau, B^\tau) = \mathbf{z}^I$, where $I \in \mathbb{Z}_{\geq 0}^n$.*

Lemma 6 (Lemma 12 of [7]). *Let Ω be an $n \times n$ matrix with nonnegative integer entries which has the property (I). Let $R(\mathbf{z})$ be a nonzero polynomial in $C[z_1, \dots, z_n]$. If $R(\Omega \mathbf{z})$ divides $R(\mathbf{z})\mathbf{z}^I$, where $I \in \mathbb{Z}_{\geq 0}^n$, then $R(\mathbf{z})$ is a monomial in z_1, \dots, z_n .*

Lemma 7 (Lemma 6 of [8]). *Let $P(\mathbf{z})$ be a nonconstant polynomial in $\mathbf{z} = (z_1, \dots, z_n)$ with $n \geq 2$. Let Ω be an $n \times n$ matrix with positive integer entries which has the property (I). Then*

$$\deg_{\mathbf{z}} P(\Omega \mathbf{z}) > \deg_{\mathbf{z}} P(\mathbf{z}).$$

3 Proof of the main theorem

We prove only Theorem 1, since Theorem 2 is proved in the same way.

Proof of Theorem 1. A complete set of representatives of the orbits $(\overline{\mathbb{Q}}^\times \times (\overline{\mathbb{Q}}^\times \setminus \{|q| = 1\})) / G_4$ is given by

$$\left\{ (x, q) \in (\overline{\mathbb{Q}}^\times)^2 \mid |q| < 1, 0 \leq \text{Arg } q < \pi \right\} =: \Lambda$$

since, under the action of the Klein four-group G_4 , the second component q is transformed either to q , q^{-1} , $-q$, or $-q^{-1}$. Hence it is enough to prove that the values

$$\eta_i := \Theta(x_i, q_i) = \sum_{k=1}^{\infty} \prod_{l=1}^k \frac{x_i q_i^{R_l}}{1 - q_i^{2R_l}} \quad (i = 1, \dots, r)$$

are algebraically independent for any finite number of distinct pairs (x_1, q_1) , $(x_2, q_2), \dots, (x_r, q_r)$ belonging to Λ .

Assume that the values η_1, \dots, η_r are algebraically dependent. There exist multiplicatively independent algebraic numbers β_1, \dots, β_s with $0 < |\beta_j| < 1$ ($1 \leq j \leq s$) and a primitive N -th root of unity ζ such that

$$q_i = \zeta^{m_i} \prod_{j=1}^s \beta_j^{e_{ij}} \quad (1 \leq i \leq r), \quad (8)$$

where m_1, \dots, m_s are integers with $0 \leq m_i \leq N-1$ and e_{ij} ($1 \leq i \leq r$, $1 \leq j \leq s$) are nonnegative integers (cf. Loxton and van der Poorten [4], Nishioka [5]). We can choose a positive integer p and a sufficiently large integer u , which will be determined later, such that $R_{k+p} \equiv R_k \pmod{N}$ for any $k \geq u+1$. Let $y_{j\lambda}$ ($1 \leq j \leq s$, $1 \leq \lambda \leq n$) be variables and let $\mathbf{y}_j = (y_{j1}, \dots, y_{jn})$ ($1 \leq j \leq s$), $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_s)$. Define

$$f_i(\mathbf{y}) = \sum_{k=u}^{\infty} \prod_{l=u}^k \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}_j)^{e_{ij}}}{1 - \left(\zeta^{m_i R_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}_j)^{e_{ij}} \right)^2} \quad (1 \leq i \leq r),$$

where $M(\mathbf{z})$ and Ω are defined by (5) and (4), respectively. Letting

$$\beta = (\underbrace{1, \dots, 1}_{n-1}, \beta_1, \dots, \beta_s, \underbrace{1, \dots, 1}_{n-1}),$$

we see by (7) and (8) that

$$f_i(\beta) = \sum_{k=u}^{\infty} \prod_{l=u}^k \frac{x_i q_i^{R_{l+1}}}{1 - q_i^{2R_{l+1}}} = \sum_{k=u+1}^{\infty} \prod_{l=u+1}^k \frac{x_i q_i^{R_l}}{1 - q_i^{2R_l}}$$

and so

$$\eta_i = \left(\prod_{k=1}^u \frac{x_i q_i^{R_k}}{1 - q_i^{2R_k}} \right) f_i(\beta) + \sum_{k=1}^u \prod_{l=1}^k \frac{x_i q_i^{R_l}}{1 - q_i^{2R_l}}.$$

Since η_1, \dots, η_r are algebraically dependent, so are $f_i(\beta)$ ($1 \leq i \leq r$). Let

$$\Omega' = \text{diag}(\underbrace{\Omega^p, \dots, \Omega^p}_s).$$

Then each $f_i(\mathbf{y})$ satisfies the functional equation

$$f_i(\mathbf{y}) = \left(\prod_{k=u}^{p+u-1} \frac{x_i \zeta^{m_i R_{k+1}} \prod_{j=1}^s M(\Omega^k \mathbf{y}_j)^{e_{ij}}}{1 - \left(\zeta^{m_i R_{k+1}} \prod_{j=1}^s M(\Omega^k \mathbf{y}_j)^{e_{ij}} \right)^2} \right) f_i(\Omega' \mathbf{y})$$

$$+ \sum_{k=u}^{p+u-1} \prod_{l=u}^k \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}_j)^{e_{ij}}}{1 - \left(\zeta^{m_i R_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}_j)^{e_{ij}} \right)^2},$$

where $\Omega' \mathbf{y} = (\Omega^p \mathbf{y}_1, \dots, \Omega^p \mathbf{y}_s)$. Let $D = |\det(\Omega - E)| = |\Phi(1)|$, where E is the identity matrix. Then D is a positive integer, since $\Phi(1) \neq 0$. Let $y'_{j\lambda} = y_{j\lambda}^{1/D}$ ($1 \leq j \leq s$, $1 \leq \lambda \leq n$), $\mathbf{y}'_j = (y'_{j1}, \dots, y'_{jn})$ ($1 \leq j \leq s$), and $\mathbf{y}' = (\mathbf{y}'_1, \dots, \mathbf{y}'_s)$. Noting that $\prod_{j=1}^s M((\Omega - E)^{-1} \Omega^u \mathbf{y}_j)^{e_{ij}} = \prod_{j=1}^s M(D(\Omega - E)^{-1} \Omega^u \mathbf{y}'_j)^{e_{ij}} \in \overline{\mathbb{Q}}(\mathbf{y}')$, we define

$$\begin{aligned} g_i(\mathbf{y}') &= \left(\prod_{j=1}^s M((\Omega - E)^{-1} \Omega^u \mathbf{y}_j)^{e_{ij}} \right) f_i(\mathbf{y}) - T_i(\mathbf{y}') \\ &= \left(\prod_{j=1}^s M(D(\Omega - E)^{-1} \Omega^u \mathbf{y}'_j)^{e_{ij}} \right) f_i(\mathbf{y}') - T_i(\mathbf{y}') \quad (1 \leq i \leq r), \end{aligned}$$

where

$$\begin{aligned} f_i(\mathbf{y}') &= \sum_{k=u}^{\infty} \prod_{l=u}^k \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}'_j)^{D e_{ij}}}{1 - \left(\zeta^{m_i R_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}'_j)^{D e_{ij}} \right)^2} \in \overline{\mathbb{Q}}[[\mathbf{y}']], \\ T_i(\mathbf{y}') &= \left(\prod_{j=1}^s M(D(\Omega - E)^{-1} \Omega^u \mathbf{y}'_j)^{e_{ij}} \right) \sum_{k=u}^{k_1} \prod_{l=u}^k \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}'_j)^{D e_{ij}}}{1 - \left(\zeta^{m_i R_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}'_j)^{D e_{ij}} \right)^2} \\ &\in \overline{\mathbb{Q}}(\mathbf{y}'), \end{aligned}$$

and k_1 is such a large integer that $g_i(\mathbf{y}') \in \overline{\mathbb{Q}}[[\mathbf{y}']]$ ($1 \leq i \leq r$). Since $M(D(\Omega - E)^{-1} \Omega^u \mathbf{y}'_j) \prod_{k=u}^{p+u-1} M(\Omega^k \mathbf{y}'_j)^D = M(D(\Omega - E)^{-1} \Omega^{u+p} \mathbf{y}'_j)$, each $g_i(\mathbf{y}')$ satisfies the functional equation

$$\begin{aligned} g_i(\mathbf{y}') &= \left(\prod_{k=u}^{p+u-1} \frac{x_i \zeta^{m_i R_{k+1}}}{1 - \left(\zeta^{m_i R_{k+1}} \prod_{j=1}^s M(\Omega^k \mathbf{y}'_j)^{D e_{ij}} \right)^2} \right) g_i(\Omega' \mathbf{y}') \\ &+ \left(\prod_{j=1}^s M(D(\Omega - E)^{-1} \Omega^u \mathbf{y}'_j)^{e_{ij}} \right) \sum_{k=u}^{p+u-1} \prod_{l=u}^k \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}'_j)^{D e_{ij}}}{1 - \left(\zeta^{m_i R_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}'_j)^{D e_{ij}} \right)^2} \\ &+ \left(\prod_{k=u}^{p+u-1} \frac{x_i \zeta^{m_i R_{k+1}}}{1 - \left(\zeta^{m_i R_{k+1}} \prod_{j=1}^s M(\Omega^k \mathbf{y}'_j)^{D e_{ij}} \right)^2} \right) T_i(\Omega' \mathbf{y}') - T_i(\mathbf{y}'), \end{aligned}$$

where $\Omega' \mathbf{y}' = (\Omega^p \mathbf{y}'_1, \dots, \Omega^p \mathbf{y}'_s)$. Since $f_i(\boldsymbol{\beta})$ ($1 \leq i \leq r$) are algebraically dependent, so are $g_i(\boldsymbol{\beta}')$ ($1 \leq i \leq r$), where

$$\boldsymbol{\beta}' = (\underbrace{1, \dots, 1}_{n-1}, \beta_1^{1/D}, \dots, \underbrace{1, \dots, 1}_{n-1}, \beta_s^{1/D}).$$

By Lemma 1, the matrix Ω' and $\boldsymbol{\beta}'$ have the properties (I)–(IV). By Lemma 2, the functions $g_i(\mathbf{y}')$ ($1 \leq i \leq r$) are algebraically dependent over $\overline{\mathbb{Q}}(\mathbf{y}')$.

In order to apply Lemma 3, we assert that

$$Q_{ii'}(\mathbf{y}') = \prod_{k=u}^{p+u-1} \frac{x_i \zeta^{m_i R_{k+1}} \left(1 - (\zeta^{m_i R_{k+1}} \prod_{j=1}^s M(\Omega^k \mathbf{y}'_j)^{De_{i'j}})^2\right)}{x_{i'} \zeta^{m_{i'} R_{k+1}} \left(1 - (\zeta^{m_{i'} R_{k+1}} \prod_{j=1}^s M(\Omega^k \mathbf{y}'_j)^{De_{i'j}})^2\right)}$$

$$\in H = \left\{ \frac{h(\Omega' \mathbf{y}')}{h(\mathbf{y}')} \mid h(\mathbf{y}') \in \overline{\mathbb{Q}}(\mathbf{y}') \setminus \{0\} \right\}$$

if and only if $m_i = m_{i'}$, $(e_{i1}, \dots, e_{is}) = (e_{i'1}, \dots, e_{i's})$, and $x_i^p = x_{i'}^p$. It is clear that, if $m_i = m_{i'}$, $(e_{i1}, \dots, e_{is}) = (e_{i'1}, \dots, e_{i's})$, and $x_i^p = x_{i'}^p$, then $Q_{ii'}(\mathbf{y}') = 1 \in H$. Conversely, suppose that $Q_{ii'}(\mathbf{y}') \in H$. Then there exists an $F(\mathbf{y}') \in \overline{\mathbb{Q}}(\mathbf{y}') \setminus \{0\}$ satisfying

$$F(\mathbf{y}') = \left(\prod_{k=u}^{p+u-1} \frac{x_{i'} \zeta^{m_{i'} R_{k+1}} \left(1 - (\zeta^{m_{i'} R_{k+1}} \prod_{j=1}^s M(\Omega^k \mathbf{y}'_j)^{De_{i'j}})^2\right)}{x_i \zeta^{m_i R_{k+1}} \left(1 - (\zeta^{m_i R_{k+1}} \prod_{j=1}^s M(\Omega^k \mathbf{y}'_j)^{De_{i'j}})^2\right)} \right) F(\Omega' \mathbf{y}'). \quad (9)$$

Let P be a positive integer divisible by D and let

$$\mathbf{y}'_j = (y'_{j1}, \dots, y'_{jn}) = (z_1^{Pj/D}, \dots, z_n^{Pj/D}) \quad (1 \leq j \leq s).$$

We choose a sufficiently large P such that the following two properties are both satisfied:

- (a) If $(e_{i1}, \dots, e_{is}) \neq (e_{i'1}, \dots, e_{i's})$, then $\sum_{j=1}^s e_{ij} P^j \neq \sum_{j=1}^s e_{i'j} P^j$.
- (b) $F^*(\mathbf{z}) = F(z_1^{P/D}, \dots, z_n^{P/D}, \dots, z_1^{P^s/D}, \dots, z_n^{P^s/D}) \in \overline{\mathbb{Q}}(z_1, \dots, z_n) \setminus \{0\}$.

Then by (9), $F^*(\mathbf{z})$ satisfies the functional equation

$$F^*(\mathbf{z}) = \left(\prod_{k=u}^{p+u-1} \frac{x_{i'} \zeta^{m_{i'} R_{k+1}} \left(1 - (\zeta^{m_{i'} R_{k+1}} M(\Omega^k \mathbf{z})^{\ell_i})^2\right)}{x_i \zeta^{m_i R_{k+1}} \left(1 - (\zeta^{m_i R_{k+1}} M(\Omega^k \mathbf{z})^{\ell_{i'}})^2\right)} \right) F^*(\Omega^p \mathbf{z}), \quad (10)$$

where $\ell_i = \sum_{j=1}^s e_{ij} P^j$ ($1 \leq i \leq r$). Therefore by Lemma 4 we see that

$$\frac{x_{i'} \zeta^{m_{i'} R_{k+1}} (1 - \zeta^{2m_{i'} R_{k+1}} X^{2\ell_{i'}})}{x_i \zeta^{m_i R_{k+1}} (1 - \zeta^{2m_i R_{k+1}} X^{2\ell_i})} \in \overline{\mathbb{Q}}^\times$$

for any k ($u \leq k \leq p+u-1$), where X is a variable, and $F^*(\mathbf{z}) \in \overline{\mathbb{Q}}^\times$. Hence $\ell_i = \ell_{i'}$ and $\zeta^{2m_i R_{k+1}} = \zeta^{2m_{i'} R_{k+1}}$ ($u \leq k \leq p+u-1$). Thus $(e_{i1}, \dots, e_{is}) = (e_{i'1}, \dots, e_{i's})$ by the property (a), and $\zeta^{2m_i} = \zeta^{2m_{i'}}$ since $\text{g.c.d.}(R_k, R_{k+1}, \dots, R_{k+n-1}) = 1$ for any $k \geq 1$. Hence $q_i^2 = q_{i'}^2$ by (8) and so $q_i = q_{i'}$ since $0 \leq \text{Arg } q_i < \pi$ ($1 \leq i \leq r$). Then $m_i = m_{i'}$, and the functional equation (10) becomes $x_i^p F^*(\mathbf{z}) = x_{i'}^p F^*(\Omega^p \mathbf{z})$. Since $F^*(\mathbf{z}) \in \overline{\mathbb{Q}}^\times$, we have $x_i^p = x_{i'}^p$, and the assertion is proved.

Now let S be a maximal subset of $\{1, \dots, r\}$ such that $(x_i^p, q_i) = (x_{i'}^p, q_{i'})$ for any $i, i' \in S$, which is equivalent to $x_i^p = x_{i'}^p$, $m_i = m_{i'}$, and $(e_{i1}, \dots, e_{is}) = (e_{i'1}, \dots, e_{i's})$. Fix a $\lambda \in S$ and let $\xi = x_\lambda^p$, $m = m_\lambda$, and $e_j = e_{\lambda j}$ ($1 \leq j \leq s$). Then $x_i^p = \xi$, $m_i = m$, and $(e_{i1}, \dots, e_{is}) = (e_1, \dots, e_s)$ for any $i \in S$ and by Lemma 3 there exists a $G(\mathbf{y}') \in \overline{\mathbb{Q}}(\mathbf{y}')$ satisfying

$$\begin{aligned}
G(\mathbf{y}') &= \xi \left(\prod_{k=u}^{p+u-1} \frac{\zeta^{mR_{k+1}}}{1 - \left(\zeta^{mR_{k+1}} \prod_{j=1}^s M(\Omega^k \mathbf{y}'_j)^{De_j} \right)^2} \right) G(\Omega' \mathbf{y}') \\
&+ \left(\prod_{j=1}^s M(D(\Omega - E)^{-1} \Omega^u \mathbf{y}'_j)^{e_j} \right) \\
&\times \sum_{k=u}^{p+u-1} \left(\sum_{i \in S} c_i x_i^{k-u+1} \right) \prod_{l=u}^k \frac{\zeta^{mR_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}'_j)^{De_j}}{1 - \left(\zeta^{mR_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}'_j)^{De_j} \right)^2} \\
&+ \xi \left(\prod_{k=u}^{p+u-1} \frac{\zeta^{mR_{k+1}}}{1 - \left(\zeta^{mR_{k+1}} \prod_{j=1}^s M(\Omega^k \mathbf{y}'_j)^{De_j} \right)^2} \right) \sum_{i \in S} c_i T_i(\Omega' \mathbf{y}') - \sum_{i \in S} c_i T_i(\mathbf{y}'),
\end{aligned}$$

where c_i ($i \in S$) are algebraic numbers not all zero. Then

$$G^*(\mathbf{y}') = \left(\prod_{j=1}^s M(D(\Omega - E)^{-1} \Omega^u \mathbf{y}'_j)^{e_j} \right)^{-2} \left(G(\mathbf{y}') + \sum_{i \in S} c_i T_i(\mathbf{y}') \right) \in \overline{\mathbb{Q}}(\mathbf{y}')$$

satisfies the functional equation

$$\begin{aligned}
G^*(\mathbf{y}') &= \xi \left(\prod_{k=u}^{p+u-1} \frac{\zeta^{mR_{k+1}} \prod_{j=1}^s M(\Omega^k \mathbf{y}'_j)^{2De_j}}{1 - \left(\zeta^{mR_{k+1}} \prod_{j=1}^s M(\Omega^k \mathbf{y}'_j)^{De_j} \right)^2} \right) G^*(\Omega' \mathbf{y}') \\
&+ \frac{1}{\prod_{j=1}^s M(D(\Omega - E)^{-1} \Omega^u \mathbf{y}'_j)^{e_j}} \\
&\times \sum_{k=u}^{p+u-1} \left(\sum_{i \in S} c_i x_i^{k-u+1} \right) \prod_{l=u}^k \frac{\zeta^{mR_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}'_j)^{De_j}}{1 - \left(\zeta^{mR_{l+1}} \prod_{j=1}^s M(\Omega^l \mathbf{y}'_j)^{De_j} \right)^2}. \quad (11)
\end{aligned}$$

Let P be a positive integer and let $\mathbf{y}'_j = (y'_{j1}, \dots, y'_{jn}) = (z_1^{Pj}, \dots, z_n^{Pj})$ ($1 \leq j \leq s$). We choose a sufficiently large P such that

$$H(\mathbf{z}) = G^*(z_1^P, \dots, z_n^P, \dots, z_1^{Ps}, \dots, z_n^{Ps}) \in \overline{\mathbb{Q}}(z_1, \dots, z_n).$$

Then by (11), $H(\mathbf{z})$ satisfies the functional equation

$$\begin{aligned}
H(\mathbf{z}) &= \xi \left(\prod_{k=u}^{p+u-1} \frac{\zeta^{mR_{k+1}} M(\Omega^k \mathbf{z})^{2D\ell}}{1 - (\zeta^{mR_{k+1}} M(\Omega^k \mathbf{z})^{D\ell})^2} \right) H(\Omega^p \mathbf{z}) \\
&+ \frac{1}{M(D(\Omega - E)^{-1} \Omega^u \mathbf{z})^\ell} \sum_{k=u}^{p+u-1} \left(\sum_{i \in S} c_i x_i^{k-u+1} \right) \prod_{l=u}^k \frac{\zeta^{mR_{l+1}} M(\Omega^l \mathbf{z})^{D\ell}}{1 - (\zeta^{mR_{l+1}} M(\Omega^l \mathbf{z})^{D\ell})^2},
\end{aligned}$$

where $\ell = \sum_{j=1}^s e_j P^j$. Letting $H(\mathbf{z}) = A(\mathbf{z})/B(\mathbf{z})$, where $A(\mathbf{z})$ and $B(\mathbf{z})$ are coprime polynomials in $\overline{\mathbb{Q}}[z_1, \dots, z_n]$ with $B \not\equiv 0$, and letting $M(D(\Omega - E)^{-1} \Omega^u \mathbf{z}) = M_1(\mathbf{z})/M_2(\mathbf{z})$, where $M_1(\mathbf{z})$ and $M_2(\mathbf{z})$ are coprime monomials in $\overline{\mathbb{Q}}[z_1, \dots, z_n]$, we have

$$A(\mathbf{z})B(\Omega^p \mathbf{z})M_1(\mathbf{z})^\ell \prod_{k=u}^{p+u-1} \left(1 - (\zeta^{mR_{k+1}} M(\Omega^k \mathbf{z})^{D\ell})^2 \right)$$

$$\begin{aligned}
&= \xi A(\Omega^p \mathbf{z}) B(\mathbf{z}) M_1(\mathbf{z})^\ell \prod_{k=u}^{p+u-1} \zeta^{m R_{k+1}} M(\Omega^k \mathbf{z})^{2D\ell} \\
&\quad + \sum_{k=u}^{p+u-1} \left(\sum_{i \in S} c_i x_i^{k-u+1} \right) B(\mathbf{z}) B(\Omega^p \mathbf{z}) M_2(\mathbf{z})^\ell \prod_{l=u}^k \zeta^{m R_{l+1}} M(\Omega^l \mathbf{z})^{D\ell} \\
&\quad \times \prod_{l'=k+1}^{p+u-1} \left(1 - (\zeta^{m R_{l'+1}} M(\Omega^{l'} \mathbf{z})^{D\ell})^2 \right). \tag{12}
\end{aligned}$$

In what follows, let u be sufficiently large. By the condition $\Phi(2) < 0$, the root ρ of $\Phi(X)$ such that $R_k = b\rho^k + o(\rho^k)$ with $b > 0$ (cf. Remark 4 in [6]) satisfies $\rho > 2$ and hence $R_{k+1} > 2R_k$ for all sufficiently large k . Then the componentwise inequality $(R_n, \dots, R_1)D(\Omega - E)^{-1}\Omega^u = (R_n, \dots, R_1)\Omega^u D(\Omega - E)^{-1} = (R_{u+n}, \dots, R_{u+1})D(\Omega - E)^{-1} < D(R_{u+n}, \dots, R_{u+1})$ holds and so $z_1 \cdots z_n M_1(\mathbf{z})^\ell$ divides $M(\Omega^u \mathbf{z})^{D\ell} = M(D\Omega^u \mathbf{z})^\ell$. In what follows, p is replaced with its multiple if necessary. We can put the greatest common divisor of $A(\Omega^p \mathbf{z})$ and $B(\Omega^p \mathbf{z})$ as $\mathbf{z}^{I(p)}$, where $I(p)$ is an n -dimensional vector with nonnegative integer components, by Lemma 5. Then $B(\Omega^p \mathbf{z})$ divides $B(\mathbf{z})M_1(\mathbf{z})^\ell \mathbf{z}^{I(p)} \prod_{k=u}^{p+u-1} M(\Omega^k \mathbf{z})^{2D\ell}$ by (12). Therefore $B(\mathbf{z})$ is a monomial in z_1, \dots, z_n by Lemmas 1 and 6. Since p and u are independent, the right-hand side of (12) is divisible by $z_1 \cdots z_n M_1(\mathbf{z})^\ell B(\Omega^p \mathbf{z})$ and thus $A(\mathbf{z})$ is divisible by $z_1 \cdots z_n$. Since $A(\mathbf{z})$ and $B(\mathbf{z})$ are coprime, $B(\mathbf{z}) \in \overline{\mathbb{Q}}^\times$. If $A(\mathbf{z}) \notin \overline{\mathbb{Q}}$ and if p is sufficiently large, then by Lemma 7, $\deg_{\mathbf{z}} A(\Omega^p \mathbf{z}) > \max\{\deg_{\mathbf{z}} A(\mathbf{z}), \deg_{\mathbf{z}} M_2(\mathbf{z})^\ell\}$, which is a contradiction by comparing the total degrees of both sides of (12). Hence $A(\mathbf{z}) \in \overline{\mathbb{Q}}$. Then by (12), we see that $\sum_{i \in S} c_i x_i^{k-u+1} = 0$ ($u \leq k \leq p+u-1$) and so $\sum_{i \in S} c_i x_i^k = 0$ ($1 \leq k \leq p$). Hence $x_i = x_{i'}$ for some distinct $i, i' \in S$ since c_i ($i \in S$) are not all zero. Then $(x_i, q_i) = (x_{i'}, q_{i'})$, which is a contradiction, and the proof of the theorem is completed. \square

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